Labor Supply and Saving under Uncertainty*

Martin Flodén**

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Abstract

This paper examines how variations in labor supply can be used to self-insure against wage uncertainty, and the impact of such self-insurance on precautionary saving. The analytical framework is a two-period model with saving and labor-supply decisions where preferences are consistent with balanced growth. The main findings are that (i) labor-supply flexibility raises precautionary saving when future wages are uncertain, and (ii) uncertainty about future wages raises current labor supply and reduces future labor supply.

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**Stockholm School of Economics, Box 6501, SE-113 83 Stockholm, Sweden, phone: (+46)-8-7369263, fax: (+46)-8-313207, e-mail: Martin.Floden@hhs.se.

1 Introduction

Recent empirical and numerical studies indicate that labor supply is affected by uncertainty about future wages. For example Parker et al. (2005) find that self-employed American workers self-insure by working longer hours in response to greater uncertainty. Similarly, using a calibrated model, Low (2004) finds that young workers with much unresolved wage uncertainty work longer hours than old workers with little remaining wage uncertainty, and that labor-supply flexibility affects saving decisions over the life cycle. Low also demonstrates that allowing for labor-supply decisions and wage uncertainty is important for generating life-cycle consumption, savings, and labor supply paths that are consistent with real-world data.

In this paper, I analyze these mechanisms theoretically. More specifically, I examine how labor supply can be used to self-insure against wage uncertainty, and how labor-supply flexibility affects precautionary saving. Previous theoretical studies have typically focused either on how uncertainty affects saving in the absence of labor-supply decisions (Kimball 1990), or on how uncertainty affects labor supply in static settings without saving decisions (Eaton and Rosen 1980, Hartwick 2000, and Parker et al. 2005). To analyze how labor-supply flexibility affects saving, it is necessary to use a framework where both labor-supply and saving decisions are endogenous, but I also demonstrate that allowing for saving decisions enhances our understanding of how labor supply responds to wage uncertainty.

Eaton and Rosen (1980) showed that the effects of uncertainty on labor supply are ambiguous and that future labor supply can increase in response to increased wage uncertainty if risk aversion is sufficiently high.[1] I show here that when saving is endogenous, the tendency for wage uncertainty to reduce future labor supply is stronger, and wage uncertainty unambiguously reduces future labor supply when preferences are consistent with balanced growth. This finding is intuitive. Just as increased uncertainty tends to raise future consumption, it tends to raise future leisure. But to simultaneously raise future consumption and future leisure, it must be possible to shift resources between periods and saving cannot be ignored.

In another related paper, Bodie, Merton, and Samuelson (1992) analyzed how labor-supply flexibility influences investors' portfolio decisions. One of their findings is that greater flexibility induces more risk taking. One might therefore expect that greater labor-supply flexibility also would make agents less prudent. But this will typically not be the case. The analytical and numerical investigations in this paper show that labor-supply flexibility raises precautionary motives when wages are stochastic. Of course, flexibility does not reduce welfare, so expected utility is higher with flexible labor supply even if precautionary saving increases. With fixed labor supply, all effects of a negative shock must be absorbed by consumption. With flexible labor supply, hours worked can be adjusted to alleviate the effect of the shock. By the same argument we note that a certain amount of savings is less costly for agents when labor supply is flexible. Therefore, agents with flexible labor supply are willing to expose themselves to more risk but they can more easily save to self-insure against uncertainty.

This last point, that flexibility facilitates saving, resembles the Le Chatelier-Samuelson principle (Samuelson 1972). The contents of this principle is that the elasticity of demand of one variable is greater when other variables are allowed to adjust to price changes than when other variables are held fixed. In the present case, the amount of uncertainty is related to the value of saving. Here then, saving will increase more in response to increased uncertainty if labor supply is flexible, provided that this effect dominates the effect on risk tolerance.

[1] Hartwick (2000) and Parker et al. (2005) use similar static frameworks and also conclude that wage uncertainty has ambigious theoretical effects on labor supply.

The measure of prudence (Kimball 1990) is closely related to risk aversion. Kimball and Weil (2003) broke that link, and showed that both high risk aversion and high intertemporal elasticity of substitution tend to imply much prudence. The present paper illustrates this point. Agents with decreasing absolute risk aversion can insure against wage fluctuations by bringing much wealth into the risky period, and if the intertemporal elasticity of substitution is high, it is less costly to shift wealth across periods.

Recent work by Low (1999, 2004), Marcet et al. (2002), and French (2003) examines how labor-supply decisions, savings, and uncertainty interact in dynamic equilibrium models. Low's papers are particularly relevant since they illustrate quantitatively many of the mechanisms that I examine theoretically. He assumes Cobb-Douglas utility and solves numerically a life-cycle model with wage uncertainty for different values of the intertemporal elasticity. He finds that uncertainty raises labor supply of young agents. This is consistent with my findings in Section 4: more future uncertainty implies more labor supply today. Low further finds that there is more saving when labor supply is flexible rather than fixed, and he finds a U-shaped relationship between total savings and the intertemporal elasticity when labor supply is flexible. When labor supply is fixed, he finds a negative relationship between savings and the elasticity. All this is consistent with my theoretical analysis based on Cobb-Douglas utility (Section 3.3). I interpret this as a strong indication that the results derived in the simple two-period framework are relevant also for settings with more realistic dynamics.

The remainder of the paper is structured as follows. I present a two-period model with saving and labor-supply decisions in Section 2. Thereafter, in Section 3, I describe how to measure the strength of precautionary saving motives in this framework. I compare this measure to the standard measure as defined by Kimball (1990), and I use the measure of precautionary strength to examine how labor-supply flexibility affects precautionary saving in a two-period economy. I show that labor-supply flexibility typically raises precautionary saving. In Section 4, I examine how uncertainty affects incentives to work. I show that more wage uncertainty unambiguously has a positive effect on current labor supply but a negative effect on future labor supply. Section 5 concludes.

2 A two-period model

Let us consider a standard two-period model where agents in each period choose consumption, c, leisure, l, and saving, s. Assume that preferences are time-separable, captured by the instantaneous utility function $u(c,l)$ which is strictly concave, i.e. $u_{cc}, u_{ll} < 0$, $u_{cc}u_{ll} - u_{cl}^2 > 0$, and 'precautionary', i.e. $u_c, u_l, u_{ccc}, u_{lll} > 0$. Further, agents are assumed to have one unit of time to dispose of, and throughout the analysis we assume interior solutions for the leisure choice.[2] Let us also abstract from discounting and assume zero interest rates, and assume that the agent has no initial financial wealth.

The first-period wage rate is certain, $w_1 = w$, while the second-period wage rate w_2 is uncertain. The budget constraints are thus

$$c_1 = (1 - l_1) w - s, \tag{1}$$
$$c_2 = (1 - l_2) w_2 + s. \tag{2}$$

Let ε denote a second-period wage shock with mean zero and variance σ^2, and assume that $w_2 = w + \varepsilon$.

[2] The analysis reduces to the standard analysis with exogenous labor supply if leisure is at a corner solution.

Agents choose consumption and labor supply to maximize $u(c_1, l_1) + \mathrm{E}_\varepsilon u(c_2, l_2)$. Given the time-separable utility, the leisure choice in each period is a function of contemporaneous consumption and wages,

$$u_l(c_t, l_t) = w_t u_c(c_t, l_t). \tag{3}$$

This, together with (1) and (2) defines the leisure choices as functions of saving and the shock, $l_1 \equiv L_1(s)$, and $l_2 \equiv L_2(s, \varepsilon)$. The indirect first-period utility function is then

$$\bar{v}(s) = u\left[(1 - L_1(s)) w - s, L_1(s)\right],$$

and the indirect second-period utility is

$$v(s, \varepsilon) = u\left[(1 - L_2(s, \varepsilon)) w_2 + s, L_2(s, \varepsilon)\right]. \tag{4}$$

Agents choose saving in the first period, s, to maximize expected indirect utility $\mathrm{E}_\varepsilon\left[\bar{v}(s) + v(s, \varepsilon)\right]$. The first-order condition is then

$$\bar{v}_s(s) + \mathrm{E}_\varepsilon v_s(s, \varepsilon) = 0. \tag{5}$$

3 Precautionary savings

Pratt (1964) showed that $-u_{cc}/u_c$ is a good measure of absolute degree of risk aversion. While risk aversion measures how an agent's utility is affected by uncertainty, prudence and precautionary saving measure how an agent's decisions are affected by uncertainty. Leland (1968) and Sandmo (1970) first formalized this concept and showed that a positive third derivative of the utility function is crucial for obtaining precautionary saving. Kimball (1990) paralleled Pratt's analysis and showed that a good measure of the absolute degree of prudence is $-u_{ccc}/u_{cc}$.

To derive a measure of precautionary strength that can be used in our setting, let us follow Kimball (1990) and use a second-order expansion of the first-order conditions to find an approximate expression for saving. Expand $\bar{v}_s$ and v_s around $s = 0$ and $\varepsilon = 0$ in (5), and ignore high-order terms, to get

$$\bar{v}_s(s) = \bar{v}_s(0) + \bar{v}_{ss}(0)\, s,$$

and

$$\begin{aligned} \mathrm{E}_\varepsilon v_s(s, \varepsilon) &= \mathrm{E}_\varepsilon\left[v_s(0,0) + v_{ss}(0,0)\, s + v_{s\varepsilon}(0,0)\,\varepsilon + \frac{1}{2} v_{s\varepsilon\varepsilon}(0,0)\,\varepsilon^2 + v_{ss\varepsilon}(0,0)\, s\varepsilon\right] \\ &= v_s(0,0) + v_{ss}(0,0)\, s + \frac{1}{2} v_{s\varepsilon\varepsilon}(0,0)\,\sigma^2. \end{aligned}$$

Note that $s = 0$ solves (5) when $\sigma^2 = 0$ given our assumptions of no initial wealth, no discounting, and zero interest rate. It then follows that $v_s(0,0) = -\bar{v}_s(0)$, and $v_{ss}(0,0) = \bar{v}_{ss}(0)$. So if (5) is fulfilled, we have that

$$0 = \left[-v_s(0,0) + v_{ss}(0,0)\, s\right] + \left[v_s(0,0) + v_{ss}(0,0)\, s + \frac{1}{2} v_{s\varepsilon\varepsilon}(0,0)\,\sigma^2\right].$$

Define precautionary strength as $\eta = -\frac{v_{s\varepsilon\varepsilon}}{v_{ss}}$ and solve for s to get

$$s = \eta \frac{\sigma^2}{4}. \tag{6}$$

For future reference, note that by applying the Envelope theorem on (4) we get

$$v_s(s,\varepsilon) = u_c. \tag{7}$$

Define also

$$L_\varepsilon \equiv \frac{\partial L_2(s,\varepsilon)}{\partial \varepsilon}, \text{ and } L_s \equiv \frac{\partial L_2(s,\varepsilon)}{\partial s}.$$

3.1 A note on precautionary strength

We will use (6) to examine how labor supply flexibility in combination with wage uncertainty affect precautionary saving. Before that, however, let us consider how this measure of precautionary strength relates to Kimball's measure of prudence. We could have measured the precautionary strength by the equivalent variation premium ψ that solves

$$\mathrm{E}_\varepsilon V_s(0,\varepsilon) = V_s(0-\psi, 0),$$

where $V(s,\varepsilon) \equiv \bar{v}(s) + v(s,\varepsilon)$ is the indirect life-time utility. A second-order expansion of V_s around $s=0$ and $\varepsilon=0$ results in

$$\psi = -\frac{V_{s\varepsilon\varepsilon}}{V_{ss}}\frac{\sigma^2}{2} = -\frac{v_{s\varepsilon\varepsilon}}{v_{ss}}\frac{\sigma^2}{4}. \tag{8}$$

This definition of precautionary strength thus results in the same approximation of saving as the derivation leading to (6), but it deviates slightly from Kimball's (1990) definition of prudence. Kimball (1990) defines prudence from the equivalent variation premium $\hat{\psi}$ that solves

$$\mathrm{E}_\varepsilon V_s(0,\varepsilon) = V_s\left(0, 0-\hat{\psi}\right),$$

which here results in

$$\hat{\psi} = -\frac{V_{s\varepsilon\varepsilon}}{V_{s\varepsilon}}\frac{\sigma^2}{2} = -\frac{v_{s\varepsilon\varepsilon}}{v_{s\varepsilon}}\frac{\sigma^2}{2}. \tag{9}$$

The definitions of ψ and $\hat{\psi}$ result in equivalent (up to a scale factor) measures of precautionary strength if the decision variable and the stochastic variable enter additively in the value function and are measured in the same units, as for example in the cases analyzed by Kimball (1990). In the present framework, $\hat{\psi}$ would measure how much the non-stochastic wage would have to be reduced for saving decisions to be equivalent under uncertainty and certainty. The precautionary strength is intended to measure of how much decisions (saving) change in response to uncertainty. The equivalent variation premium should therefore be related to the decision variable, not to the stochastic variable. Numerical examples in the next section (see Table 1) show that Kimball's standard measure of prudence is a misleading indicator of precautionary saving in the present setting.

3.2 Fixed labor supply

Let us now return to the question of how the ability to adjust labor supply in response to wage shocks affects precautionary saving. As a benchmark, we first consider the measure of precautionary strength when labor supply is not a choice variable. Let $\bar{l}$ denote the level that would be chosen under certainty, i.e. $\bar{l} = L_1(0) = L_2(0,0)$. Since labor supply is fixed, $L_\varepsilon = L_s = 0$ by assumption, and we obtain the measure of precautionary strength

$$\eta^{\text{fix}} = -\frac{v_{s\varepsilon\varepsilon}}{v_{ss}} = \frac{-\left(1-\bar{l}\right)^2 u_{ccc}}{u_{cc}}. \tag{10}$$

3.3 Flexible labor supply

How does labor-supply flexibility affect precautionary saving? Using the Envelope condition (7) and differentiating v_s results in

$$\eta^{\text{flex}} = -\frac{v_{s\varepsilon\varepsilon}}{v_{ss}}$$
$$= -\frac{\left(1-\bar{l}-L_\varepsilon w\right)^2 u_{ccc} + 2\left(1-\bar{l}-L_\varepsilon w\right) L_\varepsilon u_{ccl} - \left(2L_\varepsilon + L_{\varepsilon\varepsilon} w\right) u_{cc} + L_\varepsilon^2 u_{cll} + L_{\varepsilon\varepsilon} u_{cl}}{\left(1-L_s w\right) u_{cc} + u_{cl} L_s}. \quad (11)$$

To to interpret this expression, we need further restrictions on the utility function, and I assume that utility belongs to the class of functions that are consistent with balanced growth and that are commonly used in macroeconomic analysis. King et al. (1988) show that consistency with balanced growth imposes the following restrictions on utility,

$$u(c,l) = \frac{c^{1-\mu}}{1-\mu} r(l) \quad (12)$$

for $0 < \mu < 1$ and $\mu > 1$, and

$$u(c,l) = \ln c + r(l) \quad (13)$$

when $\mu = 1$. When $\mu \leq 1$, r is increasing and concave, and when $\mu > 1$, r is decreasing and convex. These utility functions imply that the income and substitution effects of wage fluctuations cancel. The derivative of second-period leisure with respect to the wage shock is thus zero, $L_\varepsilon = 0$. The measure of precautionary strength (11) then reduces to

$$\eta^{\text{flex}} = \frac{-\left(1-\bar{l}\right)^2 u_{ccc}}{\left(1-L_s w\right) u_{cc} + u_{cl} L_s}. \quad (14)$$

We want to compare the intensity of precautionary motives in settings with different degrees of labor-supply flexibility. Before comparing η^{fix} and η^{flex}, we should ask if the utility function or the economic environment should be recalibrated when the degree of labor-supply flexibility changes. The parameters in the utility function are often chosen so that the degree of risk aversion gets a plausible value. The same parameter values may however result in different degrees of risk aversions when labor supply is flexible rather than fixed. In general, it is therefore not straightforward to compare utility functions under these different assumptions. But in Appendix A.1, I demonstrate that the utility functions considered here do not suffer from this problem since risk aversion is not affected by labor-supply flexibility. As we will see below, however, the intertemporal elasticity of substitution for total expenditure is not necessarily the same when labor supply is exogenous as when it is endogenous.

Whether the economic environment, in particular the variance of wage shocks, should be recalibrated when labor-supply flexibility changes depends on how we use the analysis. If we want to examine the importance of modeling labor-supply decisions and wage uncertainty rather than ignoring labor supply and calibrating income volatility, then σ^2 should be recalibrated to hold income variance constant for different model specifications. But if we want to understand how more or less labor-supply flexibility affects precautionary saving, σ^2 should not be recalibrated. Since the analysis will focus on the latter question, I hold σ^2 constant.

To understand how labor-supply flexibility affects precautionary saving, we can thus compare η^{flex} to η^{fix}. Proposition 1 shows that the measure of precautionary strength is higher when labor supply is flexible than when labor supply is fixed.

Proposition 1 *Assume that the utility function is consistent with balanced growth. Then*

$$\eta^{\text{flex}} > \eta^{\text{fix}}.$$

Proof. Use (2) to substitute for c_2 in (3) and totally differentiate to find

$$wL_s = \frac{w^2 u_{cc}}{u_{ll} + w^2 u_{cc} - wu_{cl}}. \tag{15}$$

For the additively separable utility function (13) the cross derivative is zero, $u_{cl} = 0$, and we immediately see that $0 < wL_s < 1$ which establishes that $\eta^{\text{flex}} > \eta^{\text{fix}}$. Consider now the multiplicatively separable utility function (12), and rewrite (14) as

$$\eta^{\text{flex}} = \frac{\left(1 - \bar{l}\right)^2 u_{ccc}}{(-u_{cc}) + L_s \left(wu_{cc} - u_{cl}\right)}.$$

Since the numerator and $-u_{cc}$ are positive, it is clear that $\eta^{\text{flex}} > \eta^{\text{fix}}$ if $0 > L_s \left(wu_{cc} - u_{cl}\right) > u_{cc}$.

Let us begin with the first inequality, $L_s \left(wu_{cc} - u_{cl}\right) > 0$. Using (3) and (12) in (15) we get

$$wL_s = \frac{\mu}{1 - \frac{(1-\mu) r r_{ll}}{r_l^2}}. \tag{16}$$

Since $u_c > 0$, we know that $r > 0$. The assumptions on r also guarantee that $(1 - \mu) r_{ll} < 0$, so we see that $0 < wL_s < \mu$ and $L_s > 0$. Consider now $wu_{cc} - u_{cl}$. Again using (12) and (3) we see that

$$\begin{aligned} wu_{cc} - u_{cl} &= c^{-\mu-1} \left(-w\mu r - cr_l\right) \\ &= -wrc^{-\mu-1} \\ &< 0. \end{aligned} \tag{17}$$

Since $L_s > 0$, this establishes that $L_s \left(wu_{cc} - u_{cl}\right) > 0$.

We now turn to the second inequality, $L_s \left(wu_{cc} - u_{cl}\right) > u_{cc}$. Note that $u_{cc} = -\mu rc^{-\mu-1}$ and use (16) and (17) to get

$$L_s \left(wu_{cc} - u_{cl}\right) - u_{cc} = \mu rc^{-\mu-1} \left(-\frac{wL_s}{\mu} + 1\right).$$

As we demonstrated above, $0 < wL_s/\mu < 1$. It then follows that $L_s \left(wu_{cc} - u_{cl}\right) - u_{cc} > 0$ which establishes the second inequality. ■

We have compared a setting with flexible labor supply to one with no flexibility and demonstrated that precautionary saving is larger in the former case. Does this also mean that *more flexibility* implies more precautionary saving? There is no general answer, but let us assume that the utility function is additively separable and that the intertemporal elasticity of leisure is constant,

$$u(c, l) = \ln c + \frac{b\left(l^{1-1/\gamma} - 1\right)}{1 - 1/\gamma} \tag{18}$$

where γ is the elasticity of leisure and $b = \bar{l}^{1/\gamma} / \left(1 - \bar{l}\right)$ is a constant. Part (a) of Proposition 2 establishes that precautionary saving then increases as leisure becomes more elastic in the utility function. Part (b) shows that when leisure becomes totally inelastic, saving is the same as when labor supply is fixed by regulation. The proof of this proposition is in the Appendix.

Proposition 2 *Assume that the utility function is given by (18). Then*

$$\text{(a)} \qquad \frac{\partial \eta^{\text{flex}}}{\partial \gamma} > 0$$

$$\text{(b)} \qquad \lim_{\gamma \to 0} \eta^{\text{flex}} = \eta^{\text{fix}}.$$

For other utility functions, however, more elastic labor supply does not always raise the strength of precautionary motives. Consider for example the Cobb-Douglas utility function

$$u(c, l) = \frac{\left(c^{\alpha} l^{1-\alpha}\right)^{1-1/\gamma}}{1 - 1/\gamma},$$

which is a special case of (12). With this utility function, the measure of precautionary strength is[3]

$$\eta^{\text{flex}} = \frac{\gamma \left[1 - \alpha \left(1 - 1/\gamma\right)\right] \left[2 - \alpha \left(1 - 1/\gamma\right)\right]}{w} \tag{19}$$

when labor supply is flexible, and

$$\eta^{\text{fix}} = \frac{\alpha \left[2 - \alpha \left(1 - 1/\gamma\right)\right]}{w}$$

when labor supply is fixed. Figure 1 plots the measure of precautionary strength, η, against the elasticity γ. The figure shows that the precautionary strength is higher when labor supply is flexible than when it is fixed, which is what we demonstrated analytically in Proposition 1. More interestingly, the figure displays a U-shaped relationship between the intertemporal elasticity of substitution and the precautionary strength when labor supply is flexible.

[Figure 1 here]

This U-shape is in accordance with Kimball and Weil's (2003) finding that both high intertemporal elasticity of substitution and high risk aversion imply much precautionary saving.[4] With the Cobb-Douglas utility function, absolute risk aversion against wage uncertainty is

$$r^{\text{a}} = \frac{\alpha^2/\gamma + \alpha \left(1 - \alpha\right)}{w}.$$

The intertemporal elasticity of substitution for total expenditure, $c + wl$, depends on the flexibility of labor supply and is

$$i^{\text{flex}} = \gamma$$

when labor supply is flexible, and

$$i^{\text{fix}} = \frac{\alpha^2}{w} \frac{1}{r^{\text{a}}}$$

when labor supply is fixed. Note that when labor supply is fixed, we get the standard result that the intertemporal elasticity is proportional to the inverse of risk aversion. Note also that

[3] See Appendix A.2 for calculations based on the Cobb-Douglas utility function.

[4] If risk aversion is high and if the agent has decreasing absolute risk aversion, precautionary behavior reduces the utility cost of uncertainty. If intertemporal elasticity is high, the utility cost of precautionary behavior that reallocates resources between periods is low.

the degree of risk aversion does not fall to zero as γ increases to infinity, and that the intertemporal elasticity of substitution is bounded from above when labor supply is fixed but unbounded when labor supply is flexible. When labor supply is flexible and γ is high, further increases in γ still raise the intertemporal elasticity one-for-one but only imply minor reductions in risk aversion. For sufficiently high γ, the effect of higher intertemporal substitution thus dominates over the effect from lower risk aversion. Intuitively, labor-supply flexibility facilitates intertemporal substitution, and raises precautionary saving if risk aversion is held constant. If labor supply becomes more elastic (higher γ) and risk aversion only falls marginally (as when γ is high), precautionary saving will increase.

Low (1999) estimates individual wage processes based on data from the Consumer Expenditure Survey, and uses these processes to calibrate a life-cycle model with Cobb-Douglas utility. When labor supply is fixed, he finds (his table 1 and figure 3) that a lower elasticity of substitution (i.e. higher risk aversion) raises aggregate savings, but with flexible labor supply he finds a U-shaped relation between γ and aggregate savings.[5] These findings are rationalized by the measures of precautionary strength η^{fix} and η^{flex} displayed in Figure 1, thus indicating that the results derived in the two-period model generalize to more realistic settings with many periods.

3.4 Numerical examples

When looking at Figure 1, it should be noted that the relevance of η as a measure of precautionary saving is derived under the assumption that saving is small. Therefore, as γ approaches zero or infinity, and precautionary saving increases, it is possible that η loses its connection to the amount of savings.

To evaluate the validity of the precautionary measures for non-negligible risks, I have calculated saving as predicted by these measures in conjunction with equation (6). For various amounts of wage uncertainty, I have also solved numerically for saving directly from the Euler equation (5). The results are shown in Table 1. It is clear that the measure of precautionary strength is strongly related to the actual saving chosen by agents. The level of savings is well predicted when the standard deviation of wages is ten percent. When the standard deviation is twice as high, predictions are still roughly accurate, but saving is consistently underestimated. Note also that saving is lower when labor supply is fixed rather than flexible, as we also showed theoretically for both utility functions. Consistent with Proposition 1, we also see that the difference between fixed and flexible labor supply increases as the intertemporal elasticity increases.

[Table 1 here]

Table 1 also reports the measure of prudence calculated as in (9).[6] The table clearly shows that this measure of prudence is misleading when labor supply is a choice variable even if utility is separable in consumption and leisure. Note also that the standard measure of prudence is proportional to s^{fix} when labor supply is not a choice variable. Using the standard measure of prudence is then not a problem.

[5] In the recent version of that paper (Low 2004), aggregate savings is normalized by income rather than earnings and the discount rate is recalibrated when the elasticity is changed. Because of these normalizations and recalibrations, it is not possible to compare level differences between model specifications.

[6] This results in $\hat{\psi}^{\text{flex}} = \left(1-\bar{l}\right)\sigma^2/c$ with additively separable utility and $\hat{\psi}^{\text{flex}} = \left[2-\alpha\left(1-1/\gamma\right)\right]\sigma^2/\left(2w\right)$ with Cobb-Douglas utility.

To understand why Propositions 1 and 2 do not hold for arbitrary utility functions, it may be instructive to consider a counterexample. Assume that the utility function is

$$u(c,l) = \frac{c^{1-\mu}-1}{1-\mu} + \frac{b\left(l^{1-1/\gamma}-1\right)}{1-1/\gamma}.$$

Note that this utility function becomes identical to (18) when $\mu \to 1$, but for other μ the utility function is not consistent with balanced growth. Using the same setup as in Table 1, it turns out that saving is higher when labor supply is exogenous than when labor supply is endogenous if $\mu > 2.5$ and γ is small. For high risk aversion, μ, there is a U-shaped relation between the labor-supply elasticity γ and precautionary saving when labor supply is endogenous. For small γ, an increase in the elasticity reduces saving but for larger γ, an increase in the elasticity raises saving. For this utility function we can show that

$$L_\varepsilon = \frac{(1-\mu)\,u_c}{u_{ll} + w^2 u_{cc}},$$

when L_ε is evaluated at $\varepsilon = 0$ and $s = 0$. This shows that if risk aversion is greater than unity ($\mu > 1$), the wealth effect dominates over the substitution effect so that leisure increases in response to a higher wage. Labor-supply responses then reduce consumption volatility, compared to the case with exogenous labor supply. The insurance provided by labor-supply responses may then reduce the need for precautionary saving.

4 Precautionary labor supply

The previous analysis demonstrated that more labor-supply flexibility typically raises precautionary saving. But how does labor supply respond to uncertainty? In our previous analysis the exact ways in which agents use variations in labor supply to insure against shocks are diffuse since labor supply reacts to realized wage shocks. To isolate the effects from uncertainty, let us follow Eaton and Rosen (1980), Hartwick (2000), and Parker et al. (2005) and assume that second-period labor supply must be chosen before uncertainty is resolved. Except for this new timing, the setting is the same as above. In particular, the second-period wage rate is uncertain, $w_2 = w + \varepsilon$, and agents solve

$$\max_{c_1,l_1,c_2,l_2,s} u(c_1,l_1) + \mathrm{E}_\varepsilon u(c_2,l_2),$$

subject to (1) and (2). The first-order conditions are then

$$wu_c(c_1,l_1) = u_l(c_1,l_1),$$

$$\mathrm{E}_\varepsilon\left[w_2 u_c(c_2,l_2)\right] = \mathrm{E}_\varepsilon u_l(c_2,l_2),$$

and

$$u_c(c_1,l_1) = \mathrm{E}_\varepsilon u_c(c_2,l_2).$$

Expanding these first-order conditions around $\varepsilon = 0$, $s = 0$, and $\sigma^2 = 0$, and ignoring high-order terms as before, we get

$$s = \frac{(wu_{cc} - u_{cl})\left[2u_{cc} - \left(1-\bar{l}\right)u_{ccl}\right] - (u_{ll} - wu_{cl})\left(1-\bar{l}\right)u_{ccc}}{u_{cc}u_{ll} - u_{cl}^2}\frac{\left(1-\bar{l}\right)\sigma^2}{4} \tag{20}$$

and

$$l_2 - \bar{l} = \frac{wu_{cc} - u_{cl}}{w^2 u_{cc} - 2wu_{cl} + u_{ll}} s + \frac{2u_{cc} + w\left(1-\bar{l}\right) u_{ccc} - \left(1-\bar{l}\right) u_{ccl}}{w^2 u_{cc} - 2wu_{cl} + u_{ll}} \frac{\left(1-\bar{l}\right)\sigma^2}{2}. \tag{21}$$

Proposition 3 demonstrates that both saving and second-period leisure increase with uncertainty.[7]

Proposition 3 *Assume that the utility function is consistent with balanced growth. Then, for small σ,*

(a) $\frac{\partial s}{\partial \sigma} > 0,$

(b) $\frac{\partial l_2}{\partial \sigma} > 0.$

Proof. See the Appendix.

In a similar framework but ignoring saving decisions, Eaton and Rosen (1980), Hartwick (2000), and Parker et al. (2005) found that uncertainty has ambiguous effects on second-period labor supply and leisure. In particular, Hartwick demonstrates that labor supply is unaffected by uncertainty when utility has the Cobb-Douglas form. This result is replicated here when saving is exogenous. Calculations in the proof of Proposition 3 show that

$$\frac{\partial l_2}{\partial \sigma} = \frac{wu_{cc} - u_{cl}}{w^2 u_{cc} - 2wu_{cl} + u_{ll}} \frac{\partial s}{\partial \sigma},$$

which demonstrates that labor-supply decisions are unaffected by uncertainty if saving is fixed.

Why do the results change when saving is endogenous? The intuition is clear. An increase in uncertainty has a direct precautionary effect on consumption and leisure, tending to reduce current consumption and leisure and raise future consumption and leisure. But if saving is fixed, future labor supply must increase for future consumption to increase. It is then not possible to simultaneously raise future consumption and future leisure. When we allow for saving decisions, resources can be shifted between periods and Proposition 3 demonstrates that the direct precautionary effect then prevails, i.e. more uncertainty raises current saving and labor supply and future consumption and leisure.

This precautionary effect on labor supply and leisure is supported by empirical evidence in Parker et al. (2005). Using the Panel Study of Income Dynamics, they find that wage uncertainty is an important determinant of labor supply for self-employed American males. Consistent with the results in Proposition 3, they also find that these self-employed individuals tend to work more when wage uncertainty increases.

This precautionary effect on labor supply is also found in recent numerical studies. In a calibrated life-cycle model, Low (2004) finds that individuals with flexible labor supply work hard at low ages. When they grow older and more wage uncertainty is resolved, labor supply falls. Low also demonstrates that allowing for labor supply decisions and modelling wage uncertainty is important for explaining real-world consumption, savings, and labor supply patterns.

5 Conclusions

This paper has considered precautionary behavior of agents with flexible labor supply in a simple two-period model. The main insights are that labor-supply flexibility tends to raise saving when

[7] It is clear that both c_1 and l_1 fall when saving increases.

future wages are uncertain and that future wage uncertainty tends to raise current labor supply and future leisure.

I have used an unrealistically simple model to illustrate the mechanisms behind precautionary saving and thus a number of important questions are unanswered. Do the results extend to multi-period models? Is precautionary saving of quantitative importance? To some extent, these questions have been addressed by recent research. Whether the results apply to multi-period dynamic general equilibrium models is not clear. Huggett and Ospina (2001) argue that the existence of aggregate precautionary savings need not depend on the properties of the utility function in such models. Their finding thus indicates that the results do not extend to multi-period models. But Huggett and Ospina's findings have to be interpreted carefully. First, they do not say that the properties of the utility function are unimportant for the *magnitude* of savings. Second, their results only apply to economies with potentially binding liquidity constraints. In models with no such constraints, for example Wang (2003), the measure of prudence does matter. Third, Flodén (2005) demonstrates that it is difficult to separate precautionary savings from life-cycle savings in such models. In the setting studied by Huggett and Ospina, more uncertainty implies more income volatility. And this volatility in combination with liquidity constraints can affect life-cycle savings even if income is perfectly predictable.

The quantitative importance of precautionary savings has also been examined. Aiyagari (1994) found precautionary savings to be modest in a dynamic general equilibrium model with fixed labor supply, at least for his preferred parameterizations of income uncertainty. But recent evidence (for example Storesletten et al., 2003) indicate that income processes are substantially more volatile and persistent than assumed in the early quantitative models. In a calibrated life-cycle model, Low (1999, 2004) shows that the quantitative effects on savings and labor supply can be substantial, and the effects he finds are consistent with the theoretical predictions in the present paper.

Appendix A: Proofs and calculations

A.1 Risk aversion

To measure risk attitudes to wage uncertainty, consider an agent with wealth s in the beginning of the second period and ask how much wealth the agent is prepared to give up to avoid wage shocks, ε.[8] The risk attitude is then measured by the premium π that solves $v(s-\pi,0) = \mathrm{E}v(s,\varepsilon)$, which results in absolute risk aversion, r^{a}, being measured as

$$r^{\mathrm{a}} = -\frac{v_{\varepsilon\varepsilon}}{v_s}.$$

From the Envelope condition, we know that $v_s = u_c$. We evaluate risk aversion at $s = 0$ and $\varepsilon = 0$. Using the budget constraint and recalling that $L_\varepsilon = 0$ for the utility functions considered, we get $v_\varepsilon = \left(1-\bar{l}\right) u_c$ and $v_{\varepsilon\varepsilon} = \left(1-\bar{l}\right)^2 u_{cc}$ both when labor supply is endogenous and when labor supply is exogenous. Consequently,

$$r^{\mathrm{a}} = -\frac{\left(1-\bar{l}\right)^2 u_{cc}}{u_c}. \tag{A.1}$$

[8] This is the risk attitude to what Drèze and Modigliani (1972) call "timeless" uncertainty, i.e. the risk agents face after having chosen first period saving.

A.2 Cobb-Douglas utility

The first-order condition for leisure implies that

$$l_2 = \frac{1-\alpha}{\alpha}\frac{c_2}{w_2}. \tag{A.2}$$

From the budget constraint this in turn implies that $c_2 = \alpha(w+\varepsilon+s)$ when labor supply is flexible. The indirect second-period utility function is then

$$v(s,\varepsilon) = \frac{(w+\varepsilon+s)^{1-1/\gamma}}{1-1/\gamma}(w+\varepsilon)^{-(1-\alpha)(1-1/\gamma)}K \tag{A.3}$$

where K is a constant. Differentiate (A.3) to obtain the precautionary strength (19).

From (A.1) we calculate absolute risk aversion as

$$r^{\mathrm{a}} = -\frac{v_{\varepsilon\varepsilon}}{v_s} = \frac{\alpha^2/\gamma + \alpha(1-\alpha)}{w}.$$

As demonstrated in Appendix (A.1), this measure of risk aversion applies both to the case with fixed and flexible labor supply.

Let $x = c + wl$ denote total expenditure in a period, and let R denote the gross interest rate so that the second-period budget constraint is $x_2 = c_2 + w_2 l_2 = w_2 + Rs$. The intertemporal elasticity of substitution, i, is defined as

$$i = \frac{d(x_2/x_1)}{dR}\frac{R}{x_2/x_1}.$$

When labor supply is flexible, (A.2) and the budget constraint imply that $x = c/\alpha$. We then get $c = \alpha x$ and $l = (1-\alpha)x/w$. The Euler equation is then (ignoring uncertainty) $x_1^{-1/\gamma} = Rx_2^{-1/\gamma}$ and it is straightforward to show that $i^{\mathrm{flex}} = \gamma$. When labor supply is fixed, the Euler equation is $c_1^{\alpha(1-\mu)-1} = Rc_2^{\alpha(1-\mu)-1}$. Using $c = x - w\bar{l}$ and evaluating the elasticity at $x_1 = x_2$ we find $i^{\mathrm{fix}} = \left(x - w\bar{l}\right)/\left[x\left(1-\alpha(1-\mu)\right)\right] = \gamma/\left[1+\gamma(1/\alpha - 1)\right]$.

A.3 Proof of Proposition 2

Note that $L_\varepsilon = 0$ as argued in the text. Using this fact in (14) we get

$$\eta = \frac{-(1-l)^2 u_{ccc}}{(1-L_s w)u_{cc}} = \frac{K_1}{1-L_s w} \tag{A.4}$$

where $K_1 \equiv 2(1-l)^2/c > 0$. In these equations, only L_s depends on γ. Hence

$$\frac{\partial\eta}{\partial\gamma} = \frac{wK_1}{(1-L_s w)^2}\frac{\partial L_s}{\partial\gamma}$$

and it is clear that

$$\operatorname{sign}\left(\frac{\partial\eta}{\partial\gamma}\right) = \operatorname{sign}\left(\frac{\partial L_s}{\partial\gamma}\right).$$

Totally differentiate (3) to find the derivative of second-period leisure with respect to savings,

$$L_s = wu_{cc}/\left(u_{ll} + w^2 u_{cc}\right). \tag{A.5}$$

Consequently

$$\begin{aligned}\frac{\partial L_s}{\partial \gamma} &= -\frac{wu_{cc}}{\left(u_{ll}+w^2u_{cc}\right)^2}\frac{\partial u_{ll}}{\partial \gamma} \\ &= K_2\frac{\partial\left(-\frac{1}{\gamma}bl^{-1/\gamma-1}\right)}{\partial \gamma}\end{aligned}$$

where $K_2 \equiv -wu_{cc}/\left(u_{ll}+w^2u_{cc}\right)^2 > 0$. The coefficient b is defined as

$$b = \frac{\bar{l}^{1/\gamma}}{1-\bar{l}},$$

and since $\bar{l} = l|_{s=0,w_2=w}$ we get

$$\frac{\partial\left(-\frac{1}{\gamma}bl^{-1/\gamma-1}\right)}{\partial \gamma} = \frac{\partial\left[-\frac{1}{\gamma}l^{-1}\left(1-l\right)^{-1}\right]}{\partial \gamma} = \frac{1}{l\left(1-l\right)} > 0.$$

This shows that

$$\frac{\partial \eta}{\partial \gamma} > 0.$$

which concludes the proof of part (a).

To see that $\lim_{\gamma\to 0}\eta^{\text{flex}} = \eta^{\text{fix}}$, first note that equation (10) here yields

$$\eta^{\text{fix}} = \frac{2\left(1-l\right)^2}{c}$$

which implies that η^{fix} is invariant to γ. From (A.4) we get

$$\eta^{\text{flex}} = \frac{K_1}{1-L_s\left(\gamma\right)w},$$

and (A.5) gives

$$L_s = \frac{\gamma w/c^2}{\frac{1}{l(1-l)}+\gamma w^2/c^2}.$$

Since $0 < l < 1$, we see that $\lim_{\gamma\to 0} L_s = 0$. This verifies the Proposition,

$$\lim_{\gamma\to 0}\eta^{\text{flex}} = K_1 = \eta^{\text{fix}}.$$

■

A.4 Proof of Proposition 3

Concavity of the utility function implies that the denominator in (20) is positive. To show that $\partial s/\partial\sigma > 0$ we thus have to show that the numerator in (20) is positive, i.e. that

$$N^S \equiv \left(wu_{cc}-u_{cl}\right)\left[2u_{cc}-\left(1-\bar{l}\right)u_{ccl}\right]-\left(u_{ll}-wu_{cl}\right)\left(1-\bar{l}\right)u_{ccc} > 0.$$

Let us first consider the additively separable utility function (13). We then have

$$N^S = 2wu_{cc}^2 - u_{ll}\left(1-\bar{l}\right)u_{ccc}.$$

Since $u_{ll} < 0$ and $u_{ccc} > 0$ we see that $N^S > 0$.

Consider next the multiplicatively separable utility function (12). Since derivatives are evaluated at $\varepsilon = 0$ and $s = 0$, we have $c = \left(1 - \bar{l}\right) w$, and using the first-order condition for leisure, $wu_c = u_l$ we get $1 - \bar{l} = (1 - \mu) r / r_l$. Using this with (12) we get

$$N = \left(\frac{r_l}{1 - \mu} - \frac{r r_{ll}}{r_l} \right) c^{1-\mu} u_{ccc}.$$

We have assumed that $r_l > 0$ when $\mu < 1$ and $r_l < 0$ when $\mu > 1$ so $r_l / (1 - \mu)$ is positive. We have also assumed that $r_{ll} < 0$ when $\mu < 1$ and that $r_{ll} > 0$ when $\mu > 1$. Since $u_c > 0$ by assumption, we have implicitly assumed that $r > 0$. So $r r_{ll} / r_l < 0$, and we see that $N^S > 0$. We have thus established part (a) of the proof.

From (21) we get

$$\frac{\partial l_2}{\partial \sigma} = \frac{wu_{cc} - u_{cl}}{w^2 u_{cc} - 2wu_{cl} + u_{ll}} \frac{\partial s}{\partial \sigma} + \frac{2u_{cc} + w\left(1 - \bar{l}\right) u_{ccc} - \left(1 - \bar{l}\right) u_{ccl}}{w^2 u_{cc} - 2wu_{cl} + u_{ll}} \left(1 - \bar{l}\right) \sigma \tag{A.6}$$

Let N^L denote the numerator in the second term in (A.6). For the additively separable utility function we get

$$N^L = 2u_{cc} + cu_{ccc} = -2c^{-2} + 2cc^{-3} = 0.$$

For the multiplicatively separable utility function we get

$$\begin{aligned} N^L &= 2u_{cc} + cu_{ccc} - (1 - \mu) r r_l^{-1} u_{ccl} \\ &= -2\mu c^{-\mu-1} r + \mu (\mu + 1) cc^{-\mu-2} r + \mu (1 - \mu) r r_l^{-1} c^{-\mu-1} r_l \\ &= 0. \end{aligned}$$

The second term in (A.6) is thus zero. For the additively separable utility function we then get

$$\frac{\partial l_2}{\partial \sigma} = \frac{wu_{cc}}{w^2 u_{cc} + u_{ll}} \frac{\partial s}{\partial \sigma}$$

which is positive since $u_{cc} < 0$ and $u_{ll} < 0$. For the multiplicatively separable utility function we get

$$\frac{\partial l_2}{\partial \sigma} = \frac{-c^{-\mu} r_l (1 - \mu)^{-1}}{w^2 u_{cc} - 2wu_{cl} + u_{ll}} \frac{\partial s}{\partial \sigma}.$$

where the numerator is negative since $r_l / (1 - \mu) > 0$, and where the denominator is negative by concavity of u. This establishes that $\partial l_2 / \partial \sigma > 0$. ■

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Table 1

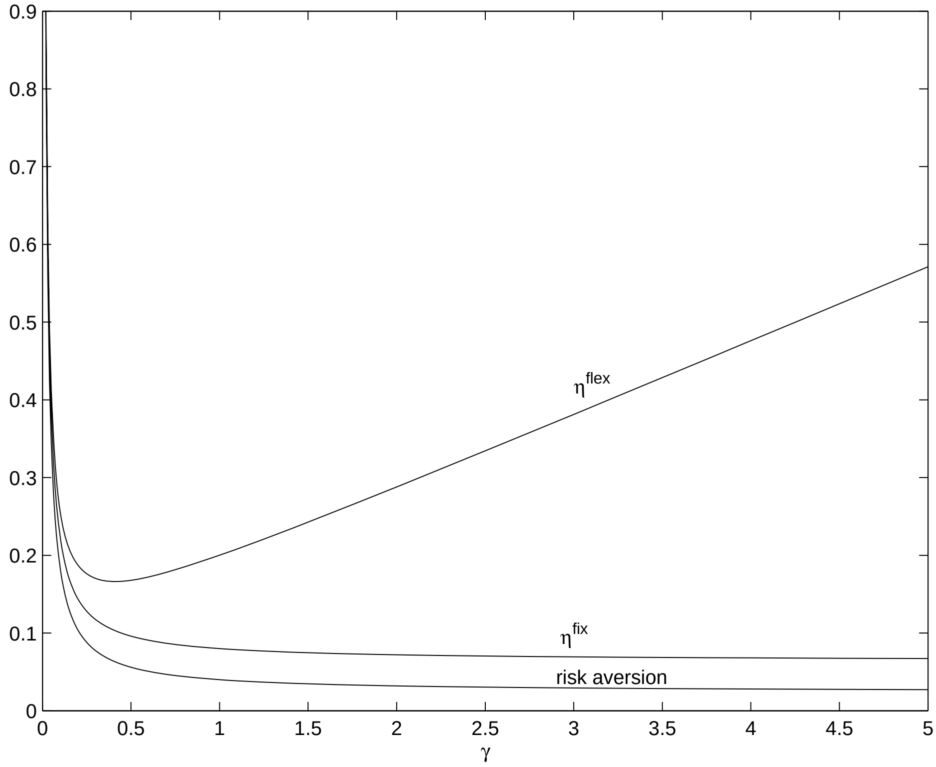
Predictive power of precautionary measure when uncertainty is not negligible

	flexible labor supply						fixed labor supply			
	$\sigma = 1$			$\sigma = 2$			$\sigma = 1$		$\sigma = 2$	
γ	s	$s(\eta)$	$\hat{\psi}$	s	$s(\eta)$	$\hat{\psi}$	s	$s(\eta)$	s	$s(\eta)$
	separable utility, $u = \ln c + \frac{b\left(l^{1-1/\gamma}-1\right)}{1-1/\gamma}$									
0.1	0.023	0.023	0.100	0.099	0.092	0.400	0.020	0.020	0.086	0.080
0.5	0.036	0.035	0.100	0.151	0.140	0.400	0.020	0.020	0.086	0.080
1.0	0.051	0.050	0.100	0.215	0.200	0.400	0.020	0.020	0.086	0.080
2.0	0.081	0.080	0.100	0.344	0.320	0.400	0.020	0.020	0.086	0.080
10.0	0.325	0.320	0.100	1.373	1.280	0.400	0.020	0.020	0.086	0.080
	Cobb-Douglas utility, $u = \frac{\left(c^{\alpha} l^{1-\alpha}\right)^{1-1/\gamma}-1}{1-1/\gamma}$									
0.1	0.067	0.064	0.280	0.322	0.258	1.120	0.058	0.056	0.288	0.224
0.5	0.043	0.042	0.120	0.183	0.168	0.480	0.024	0.024	0.104	0.096
1.0	0.051	0.050	0.100	0.215	0.200	0.400	0.020	0.020	0.086	0.080
2.0	0.073	0.072	0.090	0.308	0.288	0.360	0.018	0.018	0.077	0.072
10.0	0.266	0.262	0.082	1.118	1.050	0.328	0.017	0.016	0.070	0.066

Note: The wage process is $w = 10$ and $\varepsilon \sim N\left(0, \sigma^2\right)$. Furthermore $\alpha = 0.40$, and b is a function of γ so that labor supply equals α when $\sigma^2 = 0$.

The table compares 'true' savings, solved numerically from the first-order conditions, to the amount of savings predicted by the measures of precautionary strength. Note that $\hat{\psi} \sim s(\eta)$ when labor supply is fixed.

Figure 1
Precautionary strength with Cobb-Douglas utility



The graph relates the precautionary strength for flexible and fixed labor supply (η^{flex} and η^{fix}) to the parameter γ in the utility function

$$u(c,l) = \frac{\left(c^{\alpha} l^{1-\alpha}\right)^{1-1/\gamma} - 1}{1 - 1/\gamma},$$

where $\alpha = 0.40$.